Electromagnetism: Magnetostatics & Lorentz Force (Solutions)

FIZIKA SPhO Training

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Contents

1 Notes

In magnetostatics, we shall analyse how moving charges generate fields, which will influence the movement of other charges via the Lorentz force.

1.1 Magnetic Fields

Similar to how static charges produce electric fields, moving charges also produce magnetic fields. As a result, currents also produce their own magnetic fields, due to the movement of electrons within the wires.

We denote the **magnetic field** in a region of space as $\mathbf{B}$, for historical reasons – when Maxwell was working on electromagnetism, he assigned vector quantities in alphabetical order, and $\mathbf{B}$ was what magnetic fields were assigned! Similarly, we denote the magnetic force on a particle by $\mathbf{F}_B$, but we shall discuss the magnetic force later. The magnetic field is also sometimes called the **magnetic flux density**.

1.1.1 The Biot-Savart Law

The **Biot-Savart law** allows us to determine the magnetic field in a region of space due to a *steady current* or a *moving charge*. The law states that for a steady current I, the magnetic field at any position $\mathbf{r}$ due to the current is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{I\,\mathrm{d}\mathbf{s} \times \mathbf{r}'}{|\mathbf{r}'|^3} \tag{1}$$

Here, the integral is taken over the length of the wire, $\mathrm{d}\mathbf{s}$ is a vector on the wire pointing in the direction of conventional current of the wire, with magnitude the length of the differential element of the wire, and $\mathbf{r}'$ is the displacement vector pointing from the position of the differential element to the position of interest $\mathbf{r}$.

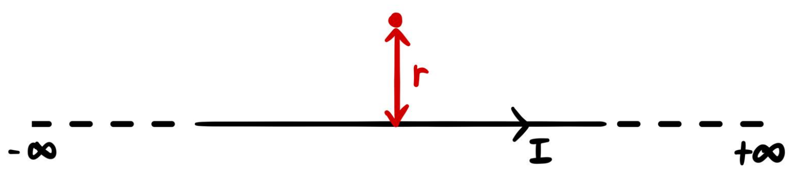
Alternatively, if we know the current density $\mathbf{J}$ at every point in space, we can also express the magnetic field as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\iiint_{\text{vol}} \frac{\mathbf{J} \times \mathbf{r}'}{|\mathbf{r}'|^3}\,\mathrm{d}V \tag{2}$$

where the integral is taken over a region of space and $\mathrm{d}V$ is a differential volume. It should not be difficult to see why the two forms are equivalent.

The use of this law is best illustrated with an example.

Example 1.1. Using the Biot-Savart law, find the magnetic field, magnitude and direction, due to an infinite straight wire carrying current I at a distance r from the wire.



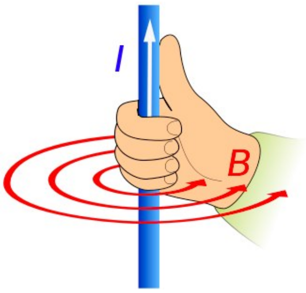
Let the wire lie on the x-axis with current pointing in the $+x$ direction, and let the point of consideration be $(0, r)$. We have that

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I\,\mathrm{d}x\,\hat{\mathbf{i}} \times \left(-x\hat{\mathbf{i}} + r\hat{\mathbf{j}}\right)}{\left|-x\hat{\mathbf{i}} + r\hat{\mathbf{j}}\right|^3} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{r\hat{\mathbf{k}}}{(r^2+x^2)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{\mu_0 I r\hat{\mathbf{k}}}{4\pi}\int_{-\infty}^{\infty} \frac{1}{(r^2+x^2)^{\frac{3}{2}}}\,\mathrm{d}x$$

The integral can be evaluated relatively easily (by a trigonometric substitution) to be equal to $\frac{2}{r^2}$, so we have

$$\mathbf{B} = \frac{\mu_0 I r \hat{\mathbf{k}}}{4\pi} \frac{2}{r^2} = \frac{\mu_0 I}{2\pi r} \hat{\mathbf{k}}$$

The magnitude of the magnetic field is $B = \frac{\mu_0 I}{2\pi r}$. To describe the direction, we can use the *right-hand rule*, which allows us to describe the direction of the magnetic field due to a current-carrying wire.



By pointing our right thumb in the direction of current, curling our remaining fingers gives us the direction that the magnetic field points. This essentially describes the cross product for us. The result we have just derived is useful to memorise as it is quite common. We will see a few more of them later.

The Biot-Savart law also states that for a moving charge q with velocity $\mathbf{v}$, the magnetic field at any position due to the moving charge is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \mathbf{r}'}{|\mathbf{r}'|^3} \tag{3}$$

Example 1.2. An electron of charge $-e$ starts at the origin at $t = 0$ and moves with speed v along the $-x$-direction. Find the magnetic field at the point $(0, y)$ as a function of time.

This is a straightforward application of the law,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{(-e)\left(-v\hat{\mathbf{i}}\right) \times \left(vt\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right)}{\left|vt\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right|^3} = \frac{\mu_0 e v y}{4\pi \left(v^2 t^2 + y^2\right)^{\frac{3}{2}}} \hat{\mathbf{k}}.$$

Note the similarity between this example and the last – the directions are the same because electrons flow in the opposite direction to conventional current.

1.1.2 Ampere's Law and Gauss's Law for Magnetism

From the Biot-Savart law, we can derive another law that allows us to evaluate the line integral of the magnetic field over some closed loop, called **Ampere's Law**. The proof requires vector calculus which is out of our scope, so we shall omit it. However, the result is very useful in determining magnetic fields, especially in situations where there is symmetry. The law states that for some closed loop C in space with some net amount of current I_{enc} passing through the loop,

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}} \tag{4}$$

Here, $\mathrm{d}\mathbf{l}$ is a differential element of C pointing along the tangent to C at its position, with magnitude the length of the differential element.

Furthermore, there is also an equivalent of **Gauss's law for magnetism**, which states that for any closed surface S in space,

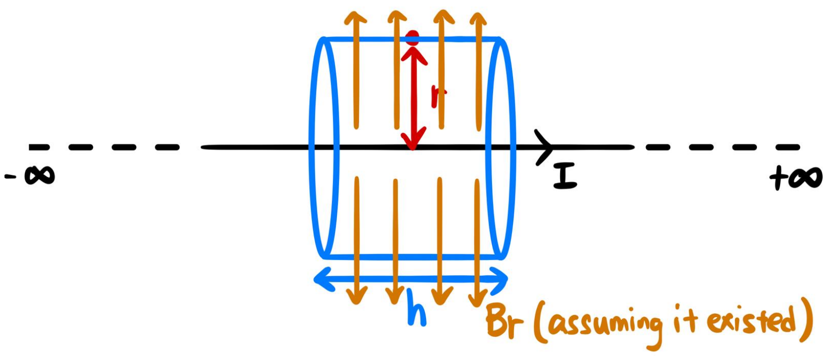
$$\oiint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \tag{5}$$

Here, d**S** is a differential element of S whose area vector is normal to the surface at its position pointing outwards, with magnitude the area of the differential element.

In most general cases, these integrals would not be very useful. However, this law can be used to greatly simplify the derivation of the magnetic fields due to some symmetric objects.

Example 1.3. Using Ampere's law and Gauss's law for magnetism, find the magnetic field, magnitude and direction, due to an infinite straight wire carrying current I at a distance r from the wire.

Let us first argue that the radial component should be zero. Consider a cylindrical Gaussian surface with radius r around the wire and height h.



Suppose the magnetic field has a radial component B_r pointing outwards. Then, by rotational symmetry, the radial component at any point around the wire at distance r will have a non-zero radial component equal to B_r. Hence, the integral evaluated over the curved surface of the Gaussian surface is

$$\iint_{\text{curved}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 2\pi r h B_r$$

since the tangential component is perpendicular to the area vector of each differential element, and thus does not contribute to the integral, and the radial component is parallel to the area vector of each differential element and is the same everywhere.

On the other hand, consider the integral over the two flat surfaces, top and bottom. By translational symmetry, the magnetic field at each point on the top surface is the same as that on the corresponding point on the bottom surface. However, since the directions of d**S** are opposite for the top and bottom surfaces, since it must point outwards, we must have that

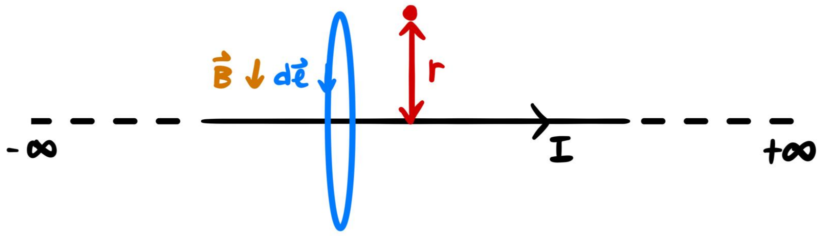
$$\iint_{\text{top}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = -\iint_{\text{bottom}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

Therefore, summing all the integrals, the integral over the entire Gaussian surface is

$$\begin{aligned}\oiint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} &= \iint_{\text{curved}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} + \iint_{\text{top}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} + \iint_{\text{bottom}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\ &= 2\pi r h B_r + 0 = 2\pi r h B_r\end{aligned}$$

Finally, by Gauss's law for magnetism, this necessitates that $B_r = 0$, since the integral is zero.

Now, we can conclude that the magnetic field points tangentially. Consider a circular loop C of radius r around the wire, where the direction of integration is dictated by the right-hand rule introduced previously, and suppose the desired (tangential) magnetic field magnitude is B.



Then, by Ampere's law,

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\text{enc}} \implies 2\pi r B = \mu_0 I \implies B = \frac{\mu_0 I}{2\pi r}$$

which retrieves our result from the previous example.

We have illustrated in this example that Ampere's law and Gauss's law for magnetism can be used in place of the Biot-Savart law in certain situations to determine the magnetic field due to certain objects. This particular example used a lot of symmetry arguments to argue the radial component of the magnetic field is zero, but it also allows us to avoid some tedious integration. Most of the time, when using Ampere's law, the intuition that currents produce magnetic fields with only tangential components help to avoid the lengthy reasoning of using Gauss's law of magnetism to show a component is zero.

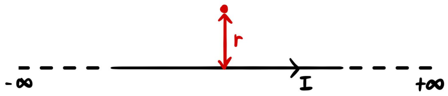
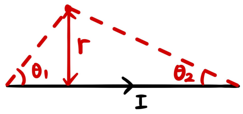
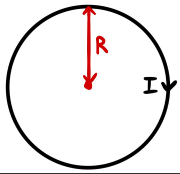
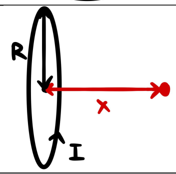
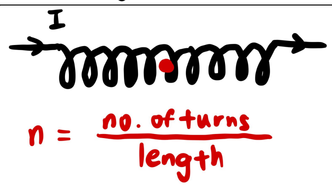
Remark. The condition for a field $\mathbf{v}$ to be conservative is that for any closed loop C,

$$\oint_C \mathbf{v} \cdot \mathrm{d}l = 0$$

Ampere's law states that, on the contrary, the integral is $\mu_0 I_{\text{enc}}$ for the magnetic field. This means that **magnetic fields are non-conservative**!

1.1.3 Common Results

There are a few common results for the magnetic fields due to various current distributions that you should aim to remember. These can be derived easily by the Biot-Savart law or Ampere's Law, which will be left as exercises to the reader.

Setup	Diagram	Field Magnitude
Infinite straight wire		$\dfrac{\mu_0 I}{2\pi r}$
Finite straight wire		$\dfrac{\mu_0 I}{4\pi r}(\cos\theta_1 + \cos\theta_2)$
Circular wire loop		$\dfrac{\mu_0 I}{2R}$
Circular wire loop		$\dfrac{\mu_0 I R^2}{2\left(R^2+x^2\right)^{\frac{3}{2}}}$
Infinite Solenoid		$\mu_0 n I$

Note that steady currents must be either closed or infinite, so a finite straight wire cannot carry steady current – but we can superimpose setups and their magnetic fields to give valid configurations.

Example 1.4. Find the magnitude of the magnetic field at the center of a square loop of side length l carrying current I.

Since there are four segments of wire which produce equal magnetic fields in the centre of the loop, applying the formula in the table above,

$$B = 4 \cdot \frac{\mu_0 I}{4\pi \frac{l}{2}}\left(\cos 45^\circ + \cos 45^\circ\right) = \frac{2\sqrt{2}\mu_0 I}{\pi l}.$$

Simple.

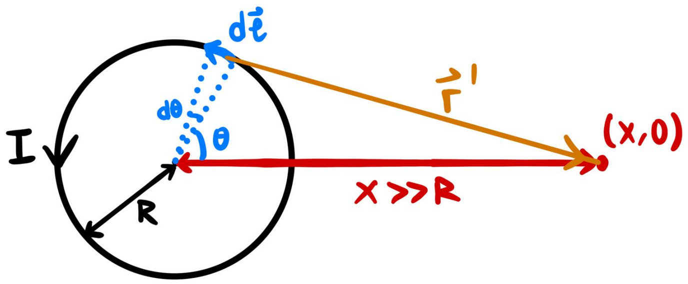
1.2 Magnetic Dipoles

An important object of study that produces magnetic fields are magnetic dipoles. Magnetic dipoles are very similar to electric dipoles, in which they carry a **magnetic dipole moment** which produces a magnetic field similar in form to an electric field. Let's begin by illustrating with an example.

1.2.1 Magnetic Dipoles are like Electric Dipoles

Example 1.5. A circular loop of wire with radius R carrying current I is placed with its centre at the origin in the xz-plane, with the current pointing anticlockwise when viewed from the $+y$-axis. Assuming that $x, y \gg R$, find the magnitude of the magnetic field at the points (i) $(x, 0)$, and (ii) $(0, y)$.

(i) We need to consider some geometry in this case, as shown in the diagram below.



Using the Biot-Savart law, we have

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I\left((R\cos\theta)\hat{\mathbf{i}}-(R\sin\theta)\hat{\mathbf{k}}\right)\mathrm{d}\theta\times\left((x-R\sin\theta)\hat{\mathbf{i}}-(R\cos\theta)\hat{\mathbf{k}}\right)}{\left|(x-R\sin\theta)\hat{\mathbf{i}}-(R\cos\theta)\hat{\mathbf{k}}\right|^3}\\ &= \frac{\mu_0 IR}{4\pi}\hat{\mathbf{j}}\int_0^{2\pi}\frac{R-x\sin\theta}{(x^2-2xR\sin\theta+R^2)^{\frac{3}{2}}}\,\mathrm{d}\theta\\ &= -\frac{\mu_0 IR}{4\pi x^2}\hat{\mathbf{j}}\int_0^{2\pi}\frac{\sin\theta-\frac{R}{x}}{\left(1-\frac{2R}{x}\sin\theta+\frac{R^2}{x^2}\right)^{\frac{3}{2}}}\,\mathrm{d}\theta\end{aligned}$$

Let $u=\frac{R}{x}$, so when $x\gg R$, $u\ll 1$. We can perform a Maclaurin series approximation of the integrand to first order in u and get

$$\begin{aligned}\frac{\sin\theta-u}{(1-2u\sin\theta+u^2)^{\frac{3}{2}}} &\approx (\sin\theta-u)\left(1+\frac{3}{2}\left(2u\sin\theta-u^2\right)\right)\\ &\approx (\sin\theta-u)\left(1+3u\sin\theta\right)\\ &\approx \sin\theta+\left(3\sin^2\theta-1\right)u\end{aligned}$$

Finally, taking the magnitude and performing the integration,

$$B_\perp\approx\frac{\mu_0 IR}{4\pi x^2}\int_0^{2\pi}\left(\sin\theta+\left(3\sin^2\theta-1\right)\frac{R}{x}\right)\mathrm{d}\theta=\frac{\mu_0 IR}{4\pi x^2}\left(0+\pi\frac{R}{x}\right)=\frac{\mu_0 IR^2}{4x^3}.$$

(ii) This case is a direct application of a formula provided previously and the use of some approximation,

$$B_\parallel=\frac{\mu_0 IR^2}{2\left(R^2+y^2\right)^{\frac{3}{2}}}\approx\frac{\mu_0 IR^2}{2y^3}.$$

since $y\gg R$.

The first part of this example is one of the more difficult examples here, requiring the use of some geometrical considerations, evaluation of cross products and a tricky first-order approximation.

1.2.2 Magnetic Dipole Moment

The previous example illustrates the similarity between a small current carrying loop of wire and an electric dipole, where the magnetic field on- and off-axis seem to be analogous to those of the electric dipole (for example, note the "factor of 2 difference" between the two expressions). Indeed, we can consider the small current carrying loop of wire to be a **magnetic dipole**, carrying some *magnetic dipole moment* **m** analogous to the electric dipole moment **p** of an electric dipole.

If we compare the expressions and consider the analogy between the fundamental constants $\frac{1}{4\pi\epsilon_0} \leftrightarrow \frac{\mu_0}{4\pi}$ as well as the dipole moments $\mathbf{p} \leftrightarrow \mathbf{m}$, we can determine that the magnetic dipole moment of this current carrying loop of wire has magnitude $m = I\pi r^2$.

In fact, for any current carrying loop of wire with area vector $\mathbf{A}$, we define the magnetic dipole moment $\mathbf{m}$ of the wire as

$$\mathbf{m} = I\mathbf{A} \tag{6}$$

Magnetic dipoles are very important objects as they also show up very often – for example, a typical bar magnet can also be considered a magnetic dipole.

We can also define the magnetic moment of a solenoid as

$$\mathbf{m} = NI\mathbf{A} \tag{7}$$

where N is the number of turns of solenoid, since the magnetic field is amplified by N times in the solenoid. Then, we can express the expressions for the magnetic field at a displacement perpendicular and parallel to the dipole respectively as

$$B_\perp = \frac{\mu_0 m}{4\pi r^3} \tag{8}$$

$$B_\parallel = \frac{\mu_0 m}{2\pi r^3} \tag{9}$$

where $m = |\mathbf{m}|$ is the magnitude of the magnetic dipole moment. These formulas would be useful to memorise.

1.2.3 Torque and Potential Energy of a Magnetic Dipole

Similar to the electric dipole, the magnetic dipole also has expressions for the torque exerted on it and its potential energy in a uniform external magnetic field $\mathbf{B}$.

We have that

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} \tag{10}$$

$$U = -\mathbf{m} \cdot \mathbf{B} \tag{11}$$

where analogous to those of the expressions for an electric dipole through $\mathbf{p} \leftrightarrow \mathbf{m}$ and $\mathbf{E} \leftrightarrow \mathbf{B}$.

It seems unusual that we can define a potential energy in the case of a magnetic dipole, since magnetic fields are non-conservative. We will see later that in this case, the magnetic force acting on the current loop is zero, and so the work done by the magnetic force to orient the dipole only depends on the magnitude of torque and angle, which makes the work depends only on the initial and final angles. This does allow us to define a potential energy, but only for magnetic dipoles.

1.3 Magnetic Forces

Now, we shall get into how magnetic fields exert forces on charges.

1.3.1 Magnetic Force

The force exerted on a charge q moving at velocity v in an external magnetic field $\mathbf{B}$ is

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} \tag{12}$$

This is the magnetic force, and it is essential to studying the dynamics of systems.

Example 1.6. A point charge q with mass m is travelling in the xz-plane at a speed of v in the $+x$-direction. When the electron reaches the origin at $t = 0$, a magnetic field B pointing in the $+y$-direction is instantaneously switched on. Determine the motion of the charge after $t = 0$.

After $t = 0$, the only force acting on the electron is the magnetic force, so using Newton's 2nd law,

$$m\mathbf{a} = q\mathbf{v} \times \mathbf{B} \qquad \Longrightarrow \qquad \mathbf{a} = \frac{qvB}{m}\left(\hat{\mathbf{v}} \times \hat{\mathbf{j}}\right)$$

To analyse the motion of the charge, we do not need to use any differential equations. Instead, note that the magnetic force is always in the xz-plane and perpendicular to the velocity (because of the cross product). This must imply that the **magnetic force does not do work** which implies that the kinetic energy, and so the speed, of the charge will remain the same. Hence, the acceleration has constant magnitude and always points perpendicular to the velocity, so the charge undergoes *circular motion*.

The initial acceleration points in the $+z$-direction, so the charge travels clockwise when viewed from the $+y$-axis. Considering the centripetal acceleration,

$$\frac{v^2}{R} = \frac{qvB}{m} \qquad \Longrightarrow \qquad R = \frac{mv}{qB} \tag{13}$$

where R is the radius of the orbit. This is often called the **cyclotron radius**. We can also calculate the angular frequency as

$$\omega = \frac{v}{R} = \frac{qB}{m} \tag{14}$$

often called the **cyclotron frequency**. These expressions are also useful to memorise.

With the expression for the magnetic force on a charged particle, we can find the magnetic force on a current carrying wire by considering the magnetic force on a moving charge in the wire

$$\mathrm{d}\mathbf{F}_B = \mathrm{d}q\,\mathbf{v} \times \mathbf{B} = I\,\mathrm{d}t\,\mathbf{v} \times \mathbf{B} = I\,\mathrm{d}\mathbf{l} \times \mathbf{B}$$

which upon integrating gives

$$\mathbf{F}_B = \int_{\text{wire}} I\,\mathrm{d}\mathbf{l} \times \mathbf{B} \tag{15}$$

In a region of uniform external magnetic field and in a wire carrying uniform current, we would then have

$$\mathbf{F}_B = I(\mathbf{l} \times \mathbf{B}) \tag{16}$$

which implies that not only does the force on a finite wire depend only on the endpoints (as they uniquely determine $\mathbf{l}$), but also that a current carrying loop of wire in a uniform external magnetic field experiences no net magnetic force.

Example 1.7. Find the net force per unit length between two parallel infinite wires separated by distance r carrying currents I_1 and I_2 in the same direction.

The magnetic field at the second wire due to the first is

$$B = \frac{\mu_0 I_1}{2\pi r}$$

so the magnetic force between the wires is

$$F_B = I_2 l B = \frac{\mu_0 I_1 I_2 l B}{2\pi r}$$

since the second wire is perpendicular to the magnetic field due to the first. Therefore, the magnetic force per unit length is

$$F_l = \frac{\mu_0 I_1 I_2 B}{2\pi r} \tag{17}$$

This is an equation that would also be useful to memorise. Analysing the cross products more carefully will result in the deduction that the magnetic force is directed towards the other wire, i.e. *like currents attract*, and similarly *unlike currents repel*.

1.3.2 Lorentz Force

The Lorentz force is the combination of both the electric and magnetic force,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{18}$$

Example 1.8. A point charge q with mass m starts from rest at the origin. At $t = 0$, an electric field E pointing in the $+x$-direction and a magnetic field B pointing in the $+y$ direction is instantaneously switched on. Determine the motion of the charge after $t = 0$.

This time, we need to solve some differential equations. Since the electric and magnetic forces lie in the xz-plane and the charge starts from rest, its motion is confined to the xz-plane. Hence, writing the equations of motion,

$$m\mathbf{a} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \Longrightarrow \quad \mathbf{a} = \frac{q}{m}\left(E\hat{\mathbf{i}} + \left(v_x\hat{\mathbf{i}} + v_z\hat{\mathbf{k}}\right) \times B\hat{\mathbf{j}}\right) = \frac{q}{m}\left((E - Bv_z)\hat{\mathbf{i}} + Bv_x\hat{\mathbf{k}}\right)$$

Separating the components, we have that

$$\dot{v}_x = \frac{q}{m}(E - Bv_z) \qquad \dot{v}_z = \frac{qB}{m}v_x$$

This set of coupled differential equations are easier to solve than it looks – taking the time derivative of the first equation gives

$$\ddot{v}_x = -\frac{qB}{m}\dot{v}_z \quad \Longrightarrow \quad \dot{v}_z = -\frac{m}{qB}\ddot{v}_x$$

which can be substituted into the second equation to give

$$-\frac{m}{qB}\ddot{v}_x = \frac{qB}{m}v_x \quad \Longrightarrow \quad \ddot{v}_x = -\frac{q^2B^2}{m^2}v_x$$

This is just the equation for a simple harmonic motion, and using the initial conditions provided gives

$$v_x = v_0 \sin\left(\frac{qB}{m}t\right)$$

for some v_0. Differentiating and substituting into the other equation,

$$\frac{qBv_0}{m}\cos\left(\frac{qB}{m}t\right) = \frac{q}{m}(E - Bv_z) \quad \Longrightarrow \quad v_z = \frac{E}{B} - v_0\cos\left(\frac{qB}{m}t\right)$$

which tells us that $v_0 = \frac{E}{B}$ from the initial conditions. Finally, integrating both equations and use the initial conditions once more gives the equations of motion,

$$x = \frac{mE}{qB^2}\left(1 - \cos\left(\frac{qB}{m}t\right)\right) \qquad z = \frac{E}{B}\left(t - \frac{m}{qB}\sin\left(\frac{qB}{m}t\right)\right)$$

There are two interesting things to note about this motion.

1. Even though the electric field points in the $+x$-direction and the magnetic field points in the $+y$-direction, as seen from the constant term in the z-component of the velocity, the charge ends up travelling in the $+z$-*direction* on average! This is called the $\mathbf{E} \times \mathbf{B}$ drift.

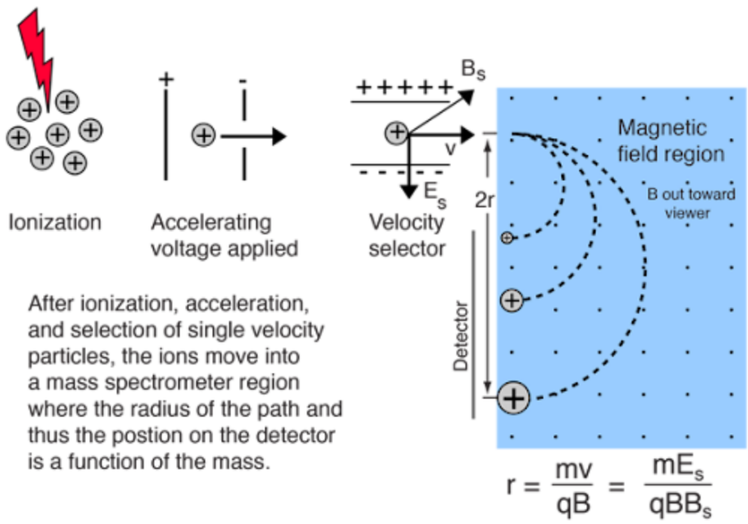
2. If we go into a frame moving at the drift speed $\frac{E}{B}$ in the $+z$-direction, the charge can be observed to travel in a circle. Although out of scope, this can be explained with the concept of field transformations. In this frame the electric field transforms to be zero, so the charge travels as if its just in a magnetic field, which we showed previously to be circular motion.

This is an example to illustrate the use of the Lorentz force. We shall now look at some applications of the Lorentz force.

1.3.3 Velocity Selectors

Velocity selectors use perpendicular electric and magnetic fields to "select" charges that are of a certain velocity. Any charges passing through the velocity selector moving at either a faster or slower speed will be deflected (in opposite directions) to either side, while charges with the desired speed will pass right through undeflected.

Velocity selectors are typically used in mass spectrometers, where charged particles are ionised, passed through a voltage and then put through the velocity selector, after which particles pass through a magnetic field. The charge to mass ratio of the particle can then be determined by the radius of the trajectory in the magnetic field.

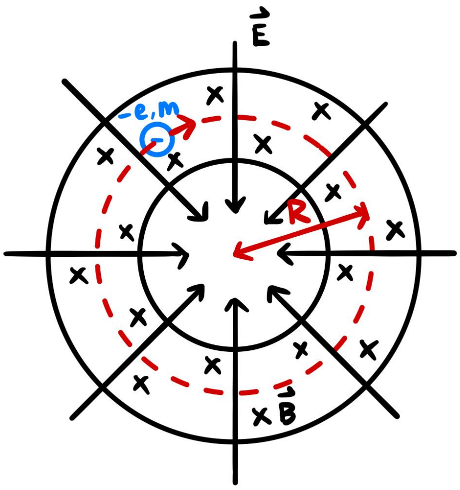


For an undeflected particle through the velocity selector, we have that the Lorentz force must be zero, so

$$q(E - vB) = 0 \quad \Longrightarrow \quad v = \frac{E}{B} \tag{19}$$

Any positive (negative) charged particle with too high of a speed will be deflected towards the positive (negative) plate, and any such particle with too low of a speed will be deflected towards the negative (positive) plate, which can be deduced by a simple analysis of the relative magnitudes of the electric and magnetic forces.

Example 1.9. A circular chamber with radius R has a radial electric field E pointing inwards and a magnetic field B perpendicular to the plane of the chamber, as shown in the diagram below. Electrons with charge $-e$ and mass m are to travel in the chamber in circular motion indefinitely. Find the minimum possible radius where this is possible.



This has the similar concept to a velocity selector, except that in order to remain in circular motion and not be deflected towards the walls of the chamber, the net force must provide the centripetal force for the charged particle to remain in circular motion. Equating the Lorentz force to the centripetal force,

$$-e(E - vB) = \frac{mv^2}{R} \quad \implies \quad \frac{mv^2}{R} - evB + eE = 0$$

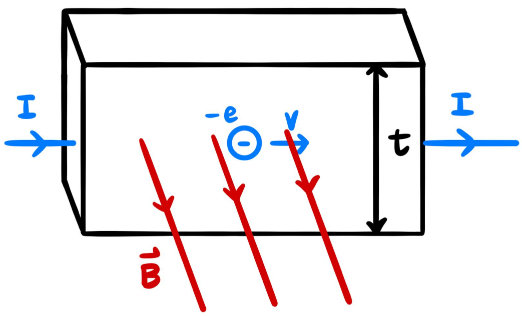
In order to travel in circular motion, some speed v must exist, so the discriminant of the quadratic equation must be at least zero. Therefore, the minimum radius occurs when the discriminant is zero,

$$e^2B^2 - \frac{4meE}{R_{\text{min}}} = 0 \quad \implies \quad R_{\text{min}} = \frac{4mE}{eB^2}.$$

1.3.4 The Hall Effect

When a current-carrying conductor is placed in a magnetic field, moving charges within the conductor experience a magnetic force due to the magnetic field, which causes them to deflect perpendicular to the current and the magnetic field. This creates a potential difference across the conductor which has an electric field. This electric field exerts an electric force on the charges that opposes the magnetic force, decreasing the rate of potential increase until the Lorentz force is zero and the system equilibrates. This is termed the **Hall Effect**.

Example 1.10. A current I passes through a conductor shaped as a cuboid with height t, as shown in the diagram below. Suppose an electron in the conductor travelling along the direction of current has drift velocity v. Find the potential difference V_{H} across the conductor, as well as which surfaces of the conductor exhibit this potential difference.



The potential difference arises from the accumulation of moving charges. As an electron travels along the direction of current, it is deflected by the magnetic field. By analysing the direction of the magnetic force, it is deflected downwards. Therefore, electrons will accumulate at the bottom surface, creating a potential difference between the *top and bottom surfaces.* Equilibrium is achieved when the electric force caused by the potential difference balances the magnetic force, and a "velocity selector" is created within the conductor where new electrons are no longer deflected.

With this information, we can then deduce that

$$v = \frac{E}{B} = \frac{V_{\mathrm{H}}}{tB} \quad \Longrightarrow \quad V_{\mathrm{H}} = tvB. \tag{20}$$

This is termed the **Hall voltage**.

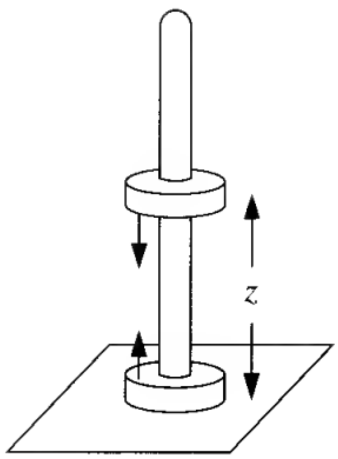
1.3.5 Magnetic Dipoles, Again

When we analyse the forces on point charges by magnetic dipoles, we can simply use the Lorentz force with the magnetic field due to the magnetic dipole, as we have illustrated with previous examples. When it comes to the interactions between dipoles, it can be a little more tricky. Instead, we can use the potential energy to our advantage, with the relationship between the force and potential energy as

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z}$$

This will allow us to calculate the interactions between magnetic dipoles.

Example 1.11. (Griffiths 6.25a) A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod (figure below). Treat the magnets as dipoles, with mass m_d and dipole moment $\mathbf{m}$. If you put two back-to-back magnets on the rod, the upper one will "float" – the magnetic force upward balancing the gravitational force downward. At what height (z) does it float?



The top magnet's position with respect to the bottom magnet's is parallel to its magnetic dipole moment. Using the previously given formulas, the magnetic field at the top magnet's position due to the bottom magnet is

$$B = \frac{\mu_0 |\mathbf{m}|}{2\pi z^3}$$

and so the magnetic potential energy is

$$U = -\mathbf{m}_{\text{top}} \cdot \mathbf{B} = \frac{\mu_0 |\mathbf{m}|^2}{2\pi z^3}$$

since the magnetic dipole moment of the top magnet and magnetic field due to the bottom magnet are antiparallel. We can then differentiate the potential energy along the z-axis to get

$$F_{\text{m}} = -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mu_0 |\mathbf{m}|^2}{2\pi z^3}\right) = \frac{3\mu_0 |\mathbf{m}|^2}{2\pi z^4}$$

This is the magnetic interaction between the two magnets. Finally, equating this to the gravitational force at equilibrium gives

$$\frac{3\mu_0 |\mathbf{m}|^2}{2\pi z^4} = m_d g \quad \Longrightarrow \quad z = \sqrt[4]{\frac{3\mu_0 |\mathbf{m}|^2}{2\pi m_d g}}.$$

1.4 Ideas

1.4.1 Conservation Laws

A very useful trick for problems involving magnetic forces are **conservation laws**. As an example, consider a point charge moving in a (possibly non-uniform and non-constant) magnetic field $\mathbf{B}$. We can compute the work done in some small time $\mathrm{d}t$ on the charge as

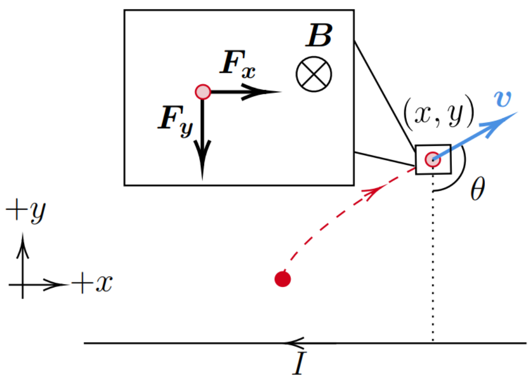
$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,\mathrm{d}t = 0$$

which arises since the cross product of two vectors is perpendicular to both of them, and so the dot product is zero. This means that in the absence of other forces, the kinetic energy of the charge is conserved by the work-energy theorem.

The example below illustrates the use of another conservation law we can construct ourselves, which we have encountered numerous times in olympiads and is **very worth understanding**.

Example 1.12. (SPhL 2024, adapted) An electron with charge $-e$ and mass m is projected with initial velocity v_0 at an initial distance y_0 from an infinitely long wire carrying current I. The direction of initial velocity is perpendicular and directed away from the wire. Determine the maximum and minimum distances y_{max} and y_{min} of the electron from the wire.

Let the wire lie along the x-axis with current pointing in the $-x$-direction, and let the electron be initially projected in the $+y$-direction.



Solving for the equations of motion of the electron will be mathematically difficult. Instead, let us first consider Newton's 2nd law in the x-direction,

$$ma_x = Bev_y \quad \Longrightarrow \quad \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mu_0 I e}{2\pi m y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

The trick that we shall use is that since we do not require any time dependence, we can multiply both sides by dt to *eliminate the time dependence*, which results in a differential equation we can solve.

$$\mathrm{d}v_x = \frac{\mu_0 I e}{2\pi m y}\,\mathrm{d}y \quad\Longrightarrow\quad v_x = \frac{\mu_0 I e}{2\pi m}\ln\left(\frac{y}{y_0}\right) \quad\Longrightarrow\quad y = y_0 \exp\left(\frac{2\pi m v_x}{\mu_0 I e}\right)$$

where we have used the initial condition that $v_x = 0$ when $y = y_0$.

Now, since the magnetic field does no work on the charge, its kinetic energy is conserved, i.e. its *speed is constant.* Therefore, at the maximum and minimum distances, where $v_y = 0$, we must have $v_x = \pm v_0$ respectively by considering the direction of the force. Hence, we can solve for the maximum and minimum distances as

$$y_{\max} = y_0 \exp\left(\frac{2\pi m v_0}{\mu_0 I e}\right),$$
$$y_{\min} = y_0 \exp\left(-\frac{2\pi m v_0}{\mu_0 I e}\right).$$

In many cases of Olympiad problems with "charges in fields" as such, we do not need to explicitly solve for the full equations of motion as a function of time. Instead, we really only concern ourselves with initial and final states (in this case, final distance), which makes conservation laws *extremely useful.* Very often they are used in conjunction with some other laws like conservation of energy and a constraint equation implied by the problem statement (in this case $v_y = 0$ at max distance from the wire).

1.4.2 Magnetic Monopoles

Previously, we talked about Gauss's law for magnetism,

$$\oiint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$$

which, comparing to Gauss's law for electricity, essentially states that there are no magnetic charges, i.e. magnetic "sources" and "sinks", and so there can never be a non-zero magnetic flux through a closed surface.

However, sometimes problems do consider hypothetical scenarios where these magnetic charges actually exist! In these cases, Gauss's law for magnetism has to be modified, since it is then possible for the magnetic flux through a closed surface to be non-zero if it encloses such a fictional magnetic charge. In such a case, a magnetic charge q_m, which has units A.m, would have a radial magnetic field

$$B = \frac{\mu_0 q_m}{4\pi r^2} \tag{21}$$

pointing outwards for a positive magnetic charge and pointing inwards for a negative magnetic charge, and in the absence of electric fields will be subjected to a force

$$F = q_m B \tag{22}$$

This is analogous to the electric field generated by an electric charge. These magnetic charges are also termed **magnetic monopoles**.

Now, if we assume the existence of hypothetical magnetic monopoles, Gauss's law for magnetism then becomes

$$\oiint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \mu_0 q_{m,\text{enc}} \tag{23}$$

In general, when dealing with magnetic monopoles, we can consider their corresponding analogies with electricity and make the appropriate substitutions. We shall see one example of an application of magnetic monopoles in the next section.

1.4.3 Gilbert Dipoles

Earlier, we introduced the idea that magnetic dipoles are like electric dipoles, and we computed the magnetic fields at a displacement perpendicular and parallel to its magnetic moment respectively by treating a magnetic dipole as a small circular current loop. This is the notion of an *Amperian dipole*, being the limit of a circular current loop as its size approaches zero.

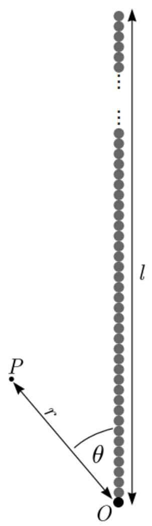
On the hand, armed with the knowledge of magnetic monopoles, we can also introduce the notion of a **Gilbert dipole**. This is analogous to that of an electric dipole, where a Gilbert dipole is comprised of two magnetic charges $\pm q_m$ separated by displacement $\mathbf{d}$ pointing from the negative to positive magnetic charge, and its magnetic dipole moment is then

$$\mathbf{m} = q_m \mathbf{d} \tag{24}$$

It is left as an exercise to rederive the previously obtained results that it does give the same magnetic field, and it is indeed analogous to an electric dipole. The Gilbert dipole thus generates the same magnetic field as an Amperian dipole outside itself. The only limitation is that the Gilbert dipole does not accurately reflect the magnetic field *within* the dipole, since it points in the wrong direction. However, it is perfectly fine to use it to determine magnetic fields outside the dipole.

This is a very powerful tool in simplifying some problems. Let's work through an example that is initially very intimidating, but can be drastically simplified using this concept.

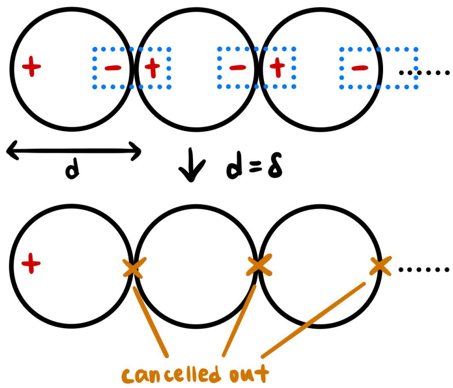
Example 1.13. (IPhO 2022 T1, adapted) Identical spherical magnets of diameter δ and magnetic dipole moment m, bound together by magnetic attraction, form a chain. Obtain an expression for the magnitude of the magnetic B-field at such a point P which is at distance r from one of the chain's endpoint O, and the angle between the chain and the line OP is θ (c.f. figure below), assuming that $l \gg r$ and $r \sin\theta \gg \delta$.



An IPhO problem? Scary! Well, we shall see that this problem is not actually as bad as it seems, using the concept of Gilbert dipoles.

Recall that we can consider a magnet as a magnetic dipole with moment m. For the spherical magnets to be bound together by attraction, their magnetic domains need to be aligned (you can verify this yourself), so they all point along the chain. Now, if we treat the magnetic dipoles as Gilbert dipoles, then we can consider each magnet to be two magnetic charges q_m separated by some distance d such that $m = q_m d$, where the two charges lie along the chain.

Here's the trick – we shall let $d = \delta$, as in the figure below.



Since $l \gg r \geq r\sin\theta \gg \delta$, it is valid for us to let $d = \delta$. When we do so, we realise that the opposite magnetic charges from adjacent magnets overlap, and thus cancel! This happens for all magnets, except for the magnets at the endpoints. Hence, we have reduced our chain of magnets to just two magnetic charges at the endpoints! Furthermore, since $l \gg r$, the magnetic field due to the magnetic charge at the other endpoint is negligible compared to the one at O.

Therefore, to compute the magnetic field at P, we only need to consider the magnetic field by the magnetic monopole at O, where we can consider to be at the centre of the magnet since $r\sin\theta \gg \delta$,

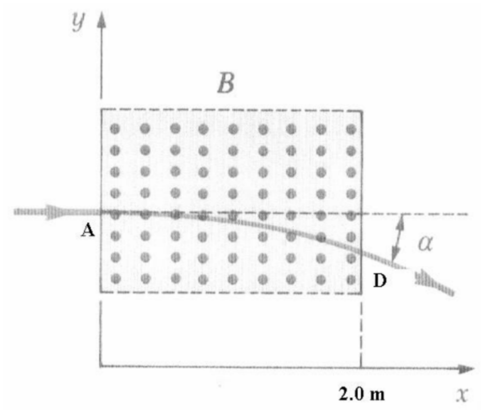
$$B = \frac{\mu_0 q_m}{4\pi r^2} = \frac{\mu_0 m}{4\pi r^2 \delta}$$

We even know the direction of the magnetic field at P – it points radially from O, either outwards or inwards, depending on the orientation of the magnetic moments of the magnets which were not specified. Amazing!

2 Problems

Problems are arranged in roughly increasing difficulty.

Problem 2.1 (SPhO 2020)**.** A beam of protons with kinetic energy 10 MeV is travelling is the positive x-direction. It enters a magnetic field of magnitude 0.15 T and the direction of the magnetic field is in the position z-direction. The magnetic field extends from $x = 0.0$ m to $x = 2.0$ m as shown below. Determine the angle α between the initial velocity vector of the proton beam and the velocity vector after the beam emerges from the magnetic field.



Solution. The speed of the beam is

$$v = \sqrt{\frac{2K}{m}} = 4.377 \times 10^7 \text{ m/s}$$

and so the radius of the circular path of the beam within the magnetic field is

$$r = \frac{mv}{eB} = 0.3045 \text{ m}.$$

Problem 2.2 (SPhO 2019)**.** In a helium dilution refrigerator, ^{3}He and ^{4}He are mixed in a special chamber to obtain extremely low temperatures. The isotopes are sent through a Bainbridge mass spectrometer. (i) The strengths of the electric and magnetic fields in the spectrometer are 100 Vm^{-1} and 0.2 T respectively. Calculate the speed of an ion which passes successfully through the velocity filter. (ii) Deduce if the spectrometer can resolve the two isotopes if the exit slit of the velocity filter is 1 mm wide.

Solution. (i) This is a velocity selector, so the speed of an undeflected ion is

$$v = \frac{E}{B} = 500 \text{ m/s}.$$

(ii) The cyclotron radius is

$$r = \frac{mv}{qB}$$

so the distance between the two isotopes after exiting the magnetic field having rotated 180° is the difference between the two diameters,

$$\Delta x = \frac{2(\Delta m)v}{qB} = 0.0256 \text{ mm}$$

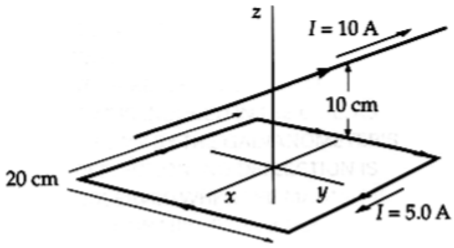
which is less than the size of the exit slit, 1 mm, so the spectrometer cannot resolve the two isotopes as they will both exit the velocity filter.

Problem 2.3. Show that the magnetic field on the axis at a point very far away from the centre of a solenoid is approximately half the magnetic field at the centre of the solenoid.

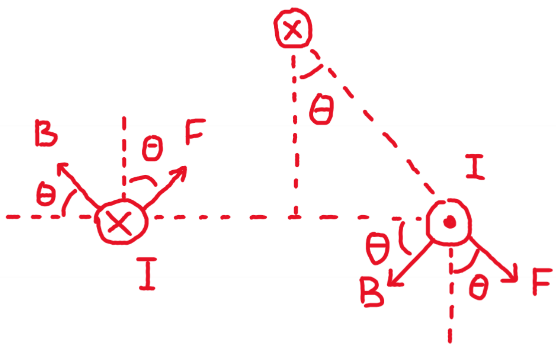
Solution. It is tempting to calculate the magnetic field directly, by considering the solenoid as a series of current loops. However, this question only requires an approximate relation, and it turns out that we only require a very simple symmetry argument to prove this fact.

The magnetic field at the center is $\mathbf{B}_{\text{center}}$ and the magnetic fields at both edges (by symmetry) are $\mathbf{B}_{\text{edge}}$. Now, suppose that we put two long solenoids of the same length together end-by-end. The magnetic field at the connected end is $2\mathbf{B}_{\text{edge}}$ by superposition. But, the combination of solenoids should effectively make a new long solenoid, and the connected end is at the center of the new solenoid, so it should also have a magnetic field that is approximately $\mathbf{B}_{\text{center}}$. Therefore, we must have $\mathbf{B}_{\text{center}} \approx 2\mathbf{B}_{\text{edge}}$, i.e. the magnetic field at the edge, very far away, is approximately half that at the center.

Problem 2.4 (Ricardo)**.** The following figure shows a square loop that has 20 cm long sides and is on the $z = 0$ plane with its center at the origin. The loop carries a current of 5.0 A. An infinitely long wire that is parallel to the x-axis and carries a current of 10 A intersects the z-axis at $z = 10$ cm. The directions of currents are shown in the figure. (a) Determine the net force acting on the loop. (b) Determine the net torque acting on the loop. (c) Explain why this torque cannot be found using $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$.



Solution. (a) It is clear that the wire segments parallel to the y-axis do not experience any force. Consider the wire segments parallel to the x-axis.



The magnetic field is

$$B = \frac{\mu_0 I_{\text{wire}}}{2\pi\sqrt{z_0^2 + \left(\frac{l_0}{2}\right)^2}}$$

and so the force acting on each segment is

$$F = I_{\text{loop}} l_0 B = \frac{\mu_0 I_{\text{wire}} I_{\text{loop}} l_0}{2\pi\sqrt{z_0^2 + \left(\frac{l_0}{2}\right)^2}}$$

Hence, the net force is

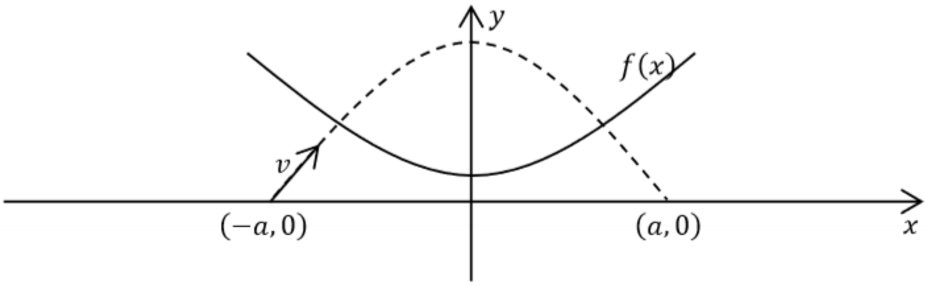
$$F_{\text{net}} = 2F\sin\theta = \frac{\mu_0 I_{\text{wire}} I_{\text{loop}} l_0^2}{2\pi\left(z_0^2 + \left(\frac{l_0}{2}\right)^2\right)} = 2.0\times 10^{-5}\ \text{N}.$$

(b) The net torque is then

$$\tau_{\text{net}} = 2F\frac{l_0}{2}\cos\theta = \frac{\mu_0 I_{\text{wire}} I_{\text{loop}} l_0^2 z_0}{2\pi\left(z_0^2 + \left(\frac{l_0}{2}\right)^2\right)} = 2.0\times 10^{-6}\ \text{N.m}.$$

(c) We cannot use $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$, although the loop is a magnetic dipole, because the loop is large. The formula only works for magnetic dipoles which can be effectively treated as a point dipole so that the magnetic field in the vicinity of the dipole can be considered uniform.

Problem 2.5 (Ricardo)**.** A point charge of charge q and mass m is moving in the xy-plane. It is launched from point $(-a,0)$ with speed v. The region above $y=f(x)$ is filled with uniform and constant magnetic field B going into the page. It is observed that no matter in which direction the initial velocity is directed, as long as the charge enters the magnetic field region, it will pass through the point $(a,0)$. (a) What is the sign of the charge? (b) With what speed does the charge pass through $(a,0)$? (c) Find the function $f(x)$.



Solution. The important fact that this question requires is that the trajectory of the point charge in the magnetic field is circular.

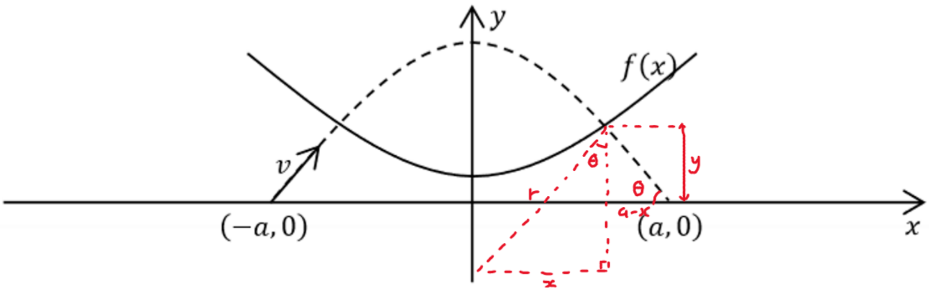
(a) Analysing the direction of the forces on the charge leads us to deduce that in order for the charge to turn clockwise, its charge must be negative.

(b) Since the magnetic field does not do any work, the kinetic energy of the charge, and so its speed, is constant. Hence, the charge also passes through $(a,0)$ with speed v.

(c) The radius of the circular trajectory in the magnetic field is

$$r = -\frac{mv}{qB}.$$

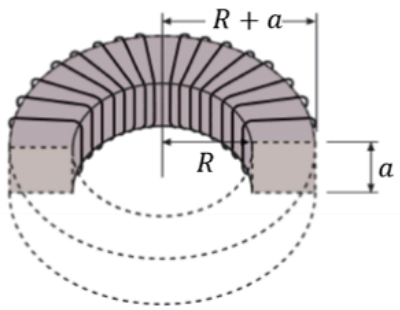
where the negative is since the charge is negative. Now, consider some geometry as shown in the diagram below.



The function $f(x)$ must also be even, since the setup is symmetric about the y-axis. Therefore, we must have

$$\frac{y}{a-|x|} = \frac{|x|}{\sqrt{r^2-x^2}} \quad \Longrightarrow \quad f(x) = \frac{a|x|-x^2}{\sqrt{\left(\frac{mv}{qB}\right)^2 - x^2}}.$$

Problem 2.6 (Ricardo)**.** A toroid with a square cross-sectional area (as shown in the picture) with dimensions $a \times a$ has N turns of wire. The current flowing is I. The inner radius is R. (a) Find the magnetic field at (i) $r < R$, (ii) $R < r < R + a$, (iii) $r > R + a$. (b) Plot the magnetic field as a function of radial distance r. (c) Show that if $a \ll R$, then the magnetic field inside the toroid is almost uniform and has a similar form to the magnetic field due to an infinitely long solenoid.



Solution. (a) Consider an Amperian loop that is a concentric circle with radius r. We have that by symmetry,

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = 2\pi r B = \mu_0 I_{\text{enc}} \quad \Longrightarrow \quad B = \frac{\mu_0 I_{\text{enc}}}{2\pi r}$$

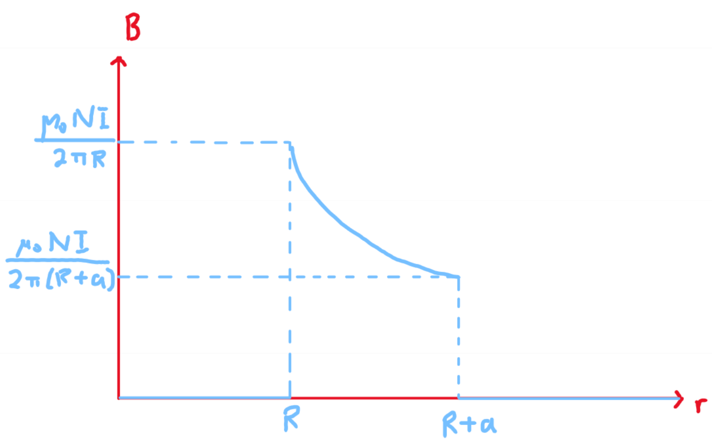
(i) For $r < R$, $I_{\text{enc}} = 0$ since the Amperian loop is completely within the hole of the toroid, so $B = 0$.

(ii) For $R < r < R + a$, $I_{\text{enc}} = NI$ since the Amperian loop contains the part of the wire at the inner radius, but not the part of the wire at the outer radius, so

$$B = \frac{\mu_0 NI}{2\pi r}.$$

(iii) For $r > R + a$, $I_{\text{enc}} = NI - NI = 0$ since the Amperian loop contains both parts of the wire carrying currents in opposite direction, so $B = 0$.

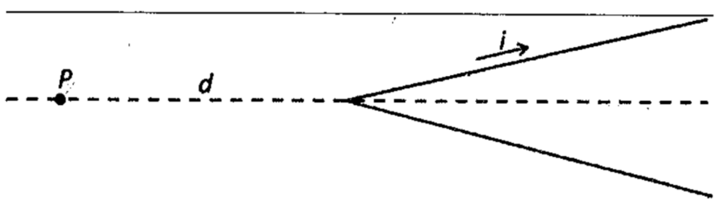
(b) Using the results of the previous part, we can plot the graph of the magnetic field as follows.



(c) If $a \ll R$, then for $R < r < R + a$, $B(r) \approx B(R)$, so the magnetic field inside the toroid is almost uniform with value

$$B = \frac{\mu_0 N I}{2\pi R} = \mu_0 n I.$$

Problem 2.7 (SPhO 2014)**.** Among the first successes of the interpretation by Ampere of magnetic phenomena, we have the computation of the magnetic field B generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart. A particularly interesting case is that of a very long thin wire, carrying a constant current i, made out of two rectlinear sections and bent in the form of a "V", with angular half-span α (see figure). According to Ampere's computations, the magnitude B of the magnetic field in a given point P lying on the axis of the "V", outside of it and at a distance d from its vertex, is proportional to $\tan\left(\frac{\alpha}{2}\right)$. Ampere's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.



Using our knowledge of electromagnetism, (a) find an expression of the magnetic field B in terms of the given quantities and any other relevant constants. Also, indicate the direction of the magnetic field. (b) Compute the field at a point P^* symmetric to P with respect to the vertex, i.e. along the axis and at the same distance d but inside the "V".

Solution. (a) Using the well-known formula for the magnetic field due to a segment of wire as well as superposition, we have that

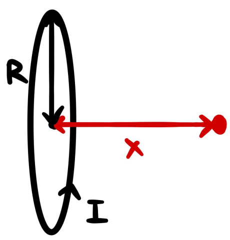
$$B = 2\frac{\mu_0 I}{4\pi r}\left(\cos\left(180^\circ - \alpha\right) + \cos 0^\circ\right) = \frac{\mu_0 I}{2\pi d}\frac{1-\cos\alpha}{\sin\alpha} = \frac{\mu_0 I}{2\pi d}\tan\frac{\alpha}{2}.$$

Here, we used the observation that the angle between P, the other end of wire at infinity and the vertex is zero, as well as a half angle identity. This formula can also be determined by the Biot-Savart's law, which we omit here.

(b) By similar argument,

$$B = 2\frac{\mu_0 I}{4\pi r}\left(\cos\alpha + \cos 0^\circ\right) = \frac{\mu_0 I}{2\pi d}\frac{1+\cos\alpha}{\sin\alpha} = \frac{\mu_0 I}{2\pi d}\cot\frac{\alpha}{2}.$$

Problem 2.8. (a) Determine the magnetic field due to a circular loop of wire of radius R, at a distance x above the centre of the loop. (b) By modelling the solenoid as many such circular loops and applying the result of part (a), show that the magnetic field in an infinite solenoid of turn density n and current I is $B = \mu_0 n I$.



Solution. (a) This is a well-known formula – we derive it here. By Biot-Savart's law, taking a cylindrical coordinate system and evaluating cross products with caution,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{I(R\,\mathrm{d}\theta)\hat{\boldsymbol{\theta}} \times (x\hat{\mathbf{z}} - R\hat{\mathbf{r}})}{|x\hat{\mathbf{z}} - R\hat{\mathbf{r}}|^3} = \frac{\mu_0 IR}{4\pi\left(R^2+x^2\right)^{\frac{3}{2}}} \int_0^{2\pi} (x\hat{\mathbf{r}} + R\hat{\mathbf{z}})\,\mathrm{d}\theta = \frac{\mu_0 IR^2}{2\left(R^2+x^2\right)^{\frac{3}{2}}}\hat{\mathbf{z}}$$

The integral obtained after simplification is a little tricky – the second term is easy because $\hat{\mathbf{z}}$ is independent of θ, but the first term is not immediately obvious since the direction of $\hat{\mathbf{r}}$ depends on θ. However, we can see that the first term evaluates zero. Consider the average value

$$\hat{\mathbf{r}}_{\text{ave}} = \frac{1}{2\pi} \int_0^{2\pi} \hat{\mathbf{r}}\,\mathrm{d}\theta$$

Clearly, this should evaluate to zero since as θ changes from 0 to 2π at constant speed, $\hat{\mathbf{r}}$ rotates around the origin at constant speed. Hence, the first term in the integral indeed evaluates to zero.

Therefore, we have that

$$B = \frac{\mu_0 IR^2}{2\left(R^2+x^2\right)^{\frac{3}{2}}}.$$

(b) Here, we neglect the winding of the solenoid and consider it as many circular loop, separated by distance $\frac{1}{n}$. Treating the position of consideration as $x = 0$, the position of the N-th loop is $\frac{N}{n}$. Hence, summing up the contributions to the magnetic field by each loop by superposition, we have

$$B = \sum_{N=-\infty}^{\infty} \frac{\mu_0 IR^2}{2\left(R^2 + \left(\frac{N}{n}\right)^2\right)^{\frac{3}{2}}}$$

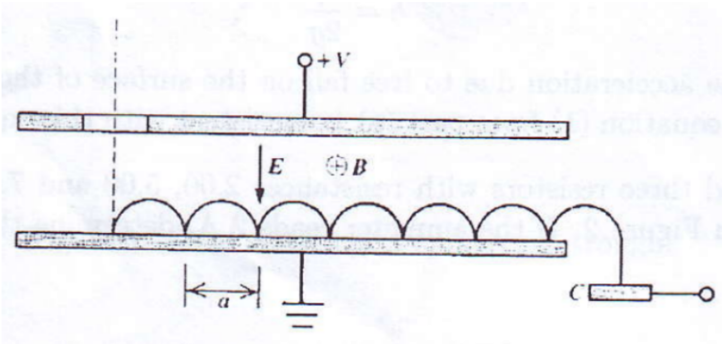
We can then approximate this sum as an integral to get

$$B \approx \int_{-\infty}^{\infty} \frac{\mu_0 IR^2}{2\left(R^2 + \left(\frac{\eta}{n}\right)^2\right)^{\frac{3}{2}}}\,\mathrm{d}\eta = \frac{\mu_0 IR^2}{2} \int_{-\infty}^{\infty} \frac{1}{\left(R^2 + \left(\frac{\eta}{n}\right)^2\right)^{\frac{3}{2}}}\,\mathrm{d}\eta$$

The integral can be evaluated easily to $\frac{2n}{R^2}$, so we have

$$B \approx \frac{\mu_0 IR^2}{2}\frac{2n}{R^2} = \mu_0 nI.$$

Problem 2.9 (SPhO 2009)**.** The figure below shows the principle of operation of a crossed-field photomultiplier. A sealed and evacuated enclosure contains two parallel plates called dynodes. These plates provide the electric field $\mathbf{E}$. An external permanent magnet superimposes a magnetic field $\mathbf{B}$. A photon ejects a low energy photoelectron. The electron accelerates upwards under the electric field but it is deflected to the negative dynode by the magnetic field. On impact with the dynode is ejects a few secondary electrons and the process repeats itself until eventually the electrons impinge on a collect C. Assuming that there is a steady voltage V between the plates and that the first photoelectron is emitted with zero initial speed, i.e. $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} = 0$ at $x = y = 0$, (i) write down differential equations to describe the horizontal and vertical velocity of the electrons as a function of x and y, (ii) solve the differential equations for $x(t)$ and $y(t)$ (this trajectory is a cycloid), (iii) find the maximum value of y, and (iv) determine the value of a.



Solution. The derivation for the equations of motion is similar to Example 1.8, except the electric and magnetic fields point in different directions. We will only quote the relevant results with the appropriate changes to the directions.

(i)
$$v_x + \left(\frac{m_e}{eB}\right)^2 \ddot{v}_x = \frac{E}{B} \qquad v_y + \left(\frac{m_e}{eB}\right)^2 \ddot{v}_y = 0$$

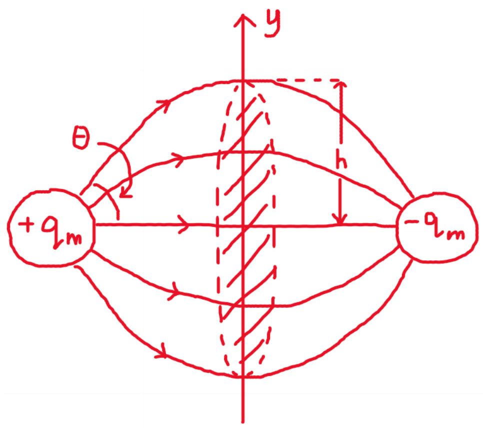
(ii)
$$x(t) = \frac{E}{B}\left(t - \frac{m_e}{eB}\sin\frac{eBt}{m_e}\right) \qquad y(t) = \frac{m_e E}{eB^2}\left(1 - \cos\frac{eBt}{m_e}\right)$$

(iii) The maximum value of y occurs when the value of the cosine in the expression is minimum, -1, so
$$y_{\max} = \frac{2m_e E}{eB^2}.$$

(iv) The value of a is the value of x after a period. The period is
$$T = \frac{2\pi m_e}{eB}$$
so the value of a is
$$a = \frac{E}{B}\left(\frac{2\pi m_e}{eB} - \frac{m_e}{eB}\sin\frac{eB\frac{2\pi m_e}{eB}}{m_e}\right) = \frac{2\pi m_e E}{eB^2}.$$

Problem 2.10. Although the field inside a Gilbert dipole does not accurately represent the internal magnetic field of a magnetic dipole, it is interesting to analyse the field within the Gilbert dipole. Consider a Gilbert dipole consisting of two magnetic charges $\pm q_m$ at $\left(\pm\frac{d}{2}, 0, 0\right)$ respectively. A magnetic field line exits the positive magnetic charge at an angle θ clockwise from the $-x$-direction, intersects the y-axis at h and enters the negative magnetic charge at an angle θ anticlockwise from the $+x$-direction. Find h.

Solution. Do not be intimidated by the presence of magnetic monopoles – this problem is effectively the same as one involving electric charges. The key to solving this problem is considering *flux*.



The field lines that pass through the circle of radius h at the origin emanate from the magnetic charge $+q_m$ subtending a half angle θ. These field lines subtend a solid angle of $2\pi(1-\cos\theta)$, so the flux through this surface is

$$\Phi_\mathrm{B} = \frac{2\pi(1-\cos\theta)}{4\pi}\mu_0 q_m = \mu_0 q_m \sin^2\frac{\theta}{2}$$

where we used a half-angle identity. On the other hand, we can compute the magnetic flux by considering the magnetic field on the yz-plane and integrating over the surface of the circle, and we get that the flux through the surface is

$$\Phi_\mathrm{B} = \int_0^h B\cdot 2\pi y\,\mathrm{d}y = \int_0^h 2\frac{\mu_0 q_m}{4\pi r^2}\frac{\frac{d}{2}}{r}(2\pi y)\,\mathrm{d}y = \frac{\mu_0 q_m d}{2}\int_0^h \frac{y}{\left(\left(\frac{d}{2}\right)^2+y^2\right)^{\frac{3}{2}}}\,\mathrm{d}y$$

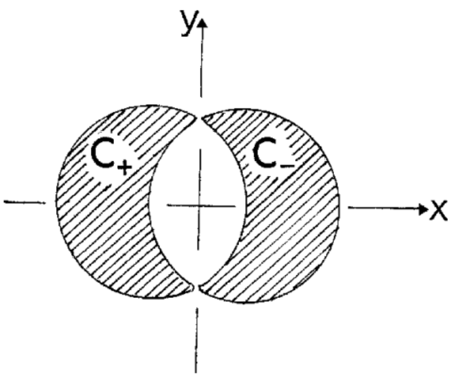
The integral evaluates easily to $2\left(\frac{1}{d}-\frac{1}{\sqrt{d^2+4h^2}}\right)$, so

$$\Phi_\mathrm{B} = \mu_0 q_m\left(1-\frac{d}{\sqrt{d^2+4h^2}}\right)$$

Equating the two expressions, we can then solve for h as

$$\mu_0 q_m \sin^2\frac{\theta}{2} = \mu_0 q_m\left(1-\frac{d}{\sqrt{d^2+4h^2}}\right) \quad\Longrightarrow\quad h = \frac{d}{2}\sqrt{\sec^4\frac{\theta}{2}-1}.$$

Problem 2.11 (IPhO 1996 T1)**.** Two straight and very long nonmagnetic conductors C_+ and C_-, insulated from each other, carry a current I in the positive and the negative z direction respectively. The cross sections of the conductors (hatched in the figure) are limited by circles of diameter D in the x-y plane, with a distance $\frac{D}{2}$ between the centres. Thereby, the resulting cross sections each have an area $\left(\frac{1}{12}\pi+\frac{1}{8}\sqrt{3}\right)D^2$. The current in each conductor is uniformly distributed over the cross section. Determine the magnetic field $B(x,y)$ in the space between the conductors.



Solution. The key to this problem is superposition, since the hollow region in the center can be considered the superposition of two opposing currents. Suppose instead there were two circular wires carrying uniform current I' in opposite directions, with centers coinciding with those of the original setup and with the same current densities. Since the current is proportional to the area for the same current density,

$$I' = \frac{\frac{\pi}{4}D^2}{\left(\frac{1}{12}\pi + \frac{1}{8}\sqrt{3}\right)D^2} I = \frac{6\pi}{2\pi + 3\sqrt{3}} I'$$

For one of the wires carrying current of magnitude I', by considering an Amperian loop that is a circle with r with center coinciding with the conductor's, the magnetic field at distance r from the center is

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = 2\pi r B = \mu_0 I_{\text{enc}} \quad \Longrightarrow \quad B = \frac{\mu_0}{2\pi r} \frac{\pi r^2 I'}{\frac{\pi}{4}D^2} = \frac{2\mu_0 I' r}{\pi D^2}$$

Then, considering the direction of the magnetic field, the magnetic field due to a circular wire carrying uniform current I' at displacement $\mathbf{r}$ from the center within the wire is

$$\mathbf{B}(\mathbf{r}) = -B\frac{y}{r}\hat{\mathbf{i}} + B\frac{x}{r}\hat{\mathbf{j}} = \frac{2\mu_0 I'}{\pi D^2}\left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right)$$

Now, superimposing both conductors and realising that the new displacements from the centers of currents $+I'$ and $-I'$ are $\mathbf{r} + \frac{D}{4}\hat{\mathbf{i}}$ and $\mathbf{r} - \frac{D}{4}\hat{\mathbf{i}}$ respectively,

$$\mathbf{B} = \frac{2\mu_0 I'}{\pi D^2}\left(\left(-y\hat{\mathbf{i}} + \left(x + \frac{D}{4}\right)\hat{\mathbf{j}}\right) - \left(-y\hat{\mathbf{i}} + \left(x - \frac{D}{4}\right)\hat{\mathbf{j}}\right)\right) = \frac{\mu_0 I'}{\pi D}\hat{\mathbf{j}}$$

Therefore, the magnetic field is uniform in the space between the conductors, and its value is

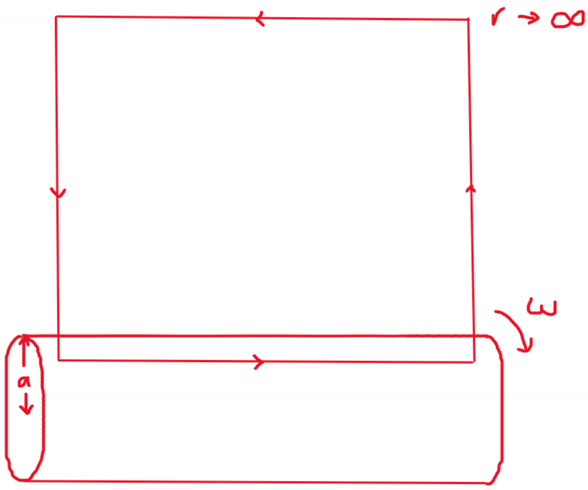
$$B(x, y) = \frac{6\mu_0 I}{\left(2\pi + 3\sqrt{3}\right)D}.$$

Problem 2.12 (SPhO 2016)**.** Two infinitely long concentric hollow cylinders have radii a and $4a$. Both cylinders and insulators; the inner cylinder has a uniformly distributed charge per length of $+\lambda$; the outer cylinder has a uniformly distributed change per length of $-\lambda$. The inner cylinder is rotating clockwise aobut its axis with an angular velocity $\omega \ll \frac{c}{a}$, where c is the speed of light. Assume that the permeability of the dielectric cylinder and the space between the cylinders is that of free space, μ_0. (a) Determine the electric field for all regions. (b) Determine the magnetic field for all regions. (c) Determine the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$, for all regions.

Solution. (a) By considering a cylindrical Gaussian surface of radius r,

$$\oiint \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 2\pi r l E = \frac{q_{\text{enc}}}{\epsilon_0} = \begin{cases} \frac{\lambda l}{\epsilon_0} & a < r < 4a \\ 0 & \text{otherwise} \end{cases} \implies E = \begin{cases} \frac{\lambda}{2\pi r \epsilon_0} & a < r < 4a, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The Amperian loop here is a little trickier. Consider a rectangular with two sides parallel to the axis of the cylinder, one of them within the cylinder and one of them infinitely far away, and with its extended plane passing through the axis of the cylinder.

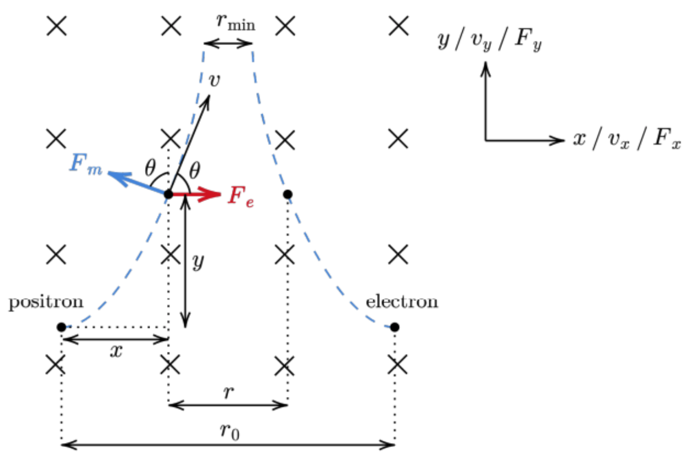


The magnetic field on the bottom side is uniform and parallel to the axis, the magnetic field on the perpendicular sides is zero as the field should be tangential, and the magnetic field at the last side is zero as it is infinitely far away. Then the only contribution to the integral is by the side in the cylinder, and

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = lB = \mu_0 I_{\text{enc}} = \begin{cases} \frac{\mu_0 \lambda l \omega}{2\pi} & r < a \\ 0 & \text{otherwise} \end{cases} \implies B = \begin{cases} \frac{\mu_0 \lambda \omega}{2\pi} & r < a, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2.13. An electron and a positron are separated by distance $r_0 = 100\ \mu$m in a region of uniform magnetic field $B = 1.00$ mT that is perpendicular to the line joining both charges. Given that the two charges are released simultaneously from rest, find the minimum distance r_{min} achieved between them throughout their motion.

Solution. The motion paths of the electron and positron are shown in the diagram below. Since both particles have equal but opposite charges, and the same mass, the forces acting on them are symmetrical, and so their paths are symmetrical.



Without loss of generality, we consider the motion of the positron. Let the positron have charge q and mass m. The two forces acting on the positron are the electrostatic force F_e and the magnetic force F_m. They are each given by:

$$F_e = \frac{kq^2}{r^2}, \quad F_m = Bqv$$

Because the magnetic force acting on the positron is always perpendicular to its velocity, the magnetic force does no work on the positron. Hence the gain in kinetic energy of the positron is only due to the loss of electric potential energy:

$$\text{EPE}_i = \text{EPE}_f + \text{KE}_f$$

$$-\frac{q^2}{4\pi\varepsilon_0 r_0} = -\frac{q^2}{4\pi\varepsilon_0 r} + 2\left(\frac{1}{2}mv^2\right) = -\frac{q^2}{4\pi\varepsilon_0 r} + m(v_x^2 + v_y^2)$$

When the distance between the positron and the electron is a minimum, $v_x = 0$. Hence, we need to find v_y, which is done by considering the force along the y-axis (due only to the magnetic force):

$$F_y = F_m \cos\theta = Bqv_x$$

$$m\frac{dv_y}{dt} = Bq\frac{dx}{dt} = Bq\frac{dx}{dt} \Rightarrow m\,dv_y = Bq\,dx$$

$$mv_y = Bqx \Rightarrow v_y = \frac{Bq}{m}\left(\frac{r_0 - r}{2}\right)$$

Therefore, we substitute the above expression for v_y to obtain:

$$-\frac{q^2}{4\pi\varepsilon_0 r_0} = -\frac{q^2}{4\pi\varepsilon_0 r} + mv_y^2$$

$$= -\frac{q^2}{4\pi\varepsilon_0 r} + \frac{B^2q^2(r_0 - r)^2}{4m}$$

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{B^2(r_0 - r)^2}{4m} \Rightarrow r^2 - r_0 r + \frac{m}{\pi\varepsilon_0 B^2 r_0} = 0$$

The solution to this quadratic equation is:

$$r = \frac{r_0 - \sqrt{r_0^2 - \frac{4m}{\pi\varepsilon_0 B^2 r_0}}}{2}$$

and we take the negative square root of the discriminant since we want the minimum distance. Substituting numerical values, the minimum distance is:

$$r_{\min} \approx \boxed{3.39\,\mu\text{m}}$$

Problem 2.14 (Kevin Zhou/Griffiths)**.** A spherical shell with radius R and uniform surface charge density σ spins with angular frequency ω about a diameter. (a) Find the magnetic field at the sphere's center. (b) Find the magnetic dipole moment of the sphere. (c) It can be shown that (1) the magnetic field inside the sphere is uniform, and (2) the magnetic field outside the sphere is exactly that of a magnetic dipole. (It requires doing some obnoxious integrals.) Using this information, make a qualitatively accurate sketch of the field. (d) There is a closely related question in electrostatics: suppose we had two solid spheres of the same radius R, with volume charge densities $\pm\rho$, and the spheres were displaced by a tiny distance $d \ll R$. Qualitatively, what would the electric field of this setup look like, for $r < R$ and $r > R$? How does it differ from your answer to part (c)?

Solution. (a) We work in cylindrical coordinates with the axis of rotation as the z-axis. Consider a small cross-sectional ring of the spherical shell at angle θ from the z-axis. Its radius is $r = R\sin\theta$ and its perpendicular distance to the center is $z = R\cos\theta$, so the current carried by this ring is

$$\mathrm{d}I = \frac{\sigma(2\pi r R\,\mathrm{d}\theta)}{T} = \sigma\omega R^2 \sin\theta\,\mathrm{d}\theta$$

Using the well-known formula for the magnetic field due to a circular loop on the axis at distance z from the center,

$$\mathrm{d}B = \frac{\mu_0\,\mathrm{d}I\,r^2}{2\left(r^2+z^2\right)^{\frac{3}{2}}} = \frac{\mu_0 R^2 \sin^2\theta}{2R^3}\left(\sigma\omega R^2\sin\theta\right)\mathrm{d}\theta = \frac{1}{2}\mu_0\sigma\omega R\sin^3\theta\,\mathrm{d}\theta$$

Integrating over the whole shell,

$$B = \frac{1}{2}\mu_0\sigma\omega R\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \frac{2}{3}\mu_0\sigma\omega R$$

pointing in the $+z$-direction, where the trigonometric integral was easily evaluated as $\frac{4}{3}$. (b) Similarly, the magnetic moment of the cross-sectional ring is

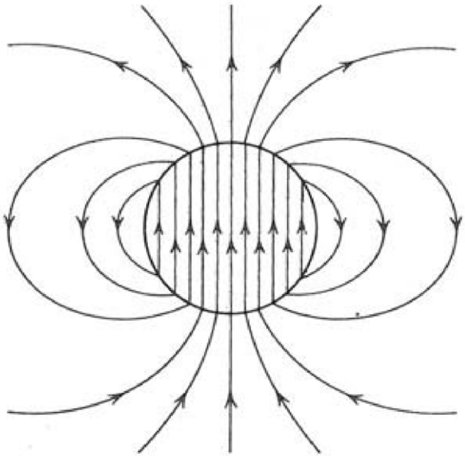
$$dm = \mathrm{d}I\,A = \left(\sigma\omega R^2\sin\theta\right)\pi(R\sin\theta)^2\,\mathrm{d}\theta = \pi\sigma\omega R^4\sin^3\theta\,\mathrm{d}\theta$$

which upon integrating over the whole sphere gives

$$m = \pi\sigma\omega R^4\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \frac{4}{3}\pi\sigma\omega R^4$$

pointing in the $+z$-direction.

(c) The magnetic field is as shown below.



The important feature is that the magnetic field lines connect at the boundary to obey Gauss's law of magnetism, i.e. the magnetic field lines form closed loops.

(d) It's easy to show by the shell theorem that the field lines will look almost identical to the previous part, where the field outside reflects that of a dipole and the field inside is uniform. However, the direction of the electric field within is in the opposite direction (since the field is conservative, so the line integral of the electric field over the closed path should be zero).

3 Advanced Problems

These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1. An infinitely long charged wire with linear charge density $\lambda = 5 \times 10^{-9}\,\mathrm{C\,m^{-1}}$ and radius $R = 0.1\,\mathrm{m}$ is placed inside a uniform magnetic field $B = 0.1\,\mathrm{T}$ directed parallel to the wire. A proton leaves the surface of the wire at an initial speed v_i in an arbitrary direction. What is the **minimum** magnitude of v_i required such that the **maximum** distance from the surface of the wire that the proton can reach is also R?

Solution. This problem neatly requires deriving a conservation law that from Newton's 2nd Law, but not in the usual sense. From Gauss' Law, we can find the electric field and potential at a distance r from the wire ($r > R$):

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad V(r) = -\int_{r_0}^{r} E(r)\,dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

where r_0 is the reference point with zero potential. By conservation of energy, the velocity of the particle must hence change as it travels. We consider each of its perpendicular components separately.

We notice that the axial component of the proton's velocity is parallel to the magnetic field and perpendicular to the electric field. Hence, it does not experience any forces due to this component. This component thus does not affect the maximum distance the proton can reach. To minimise the proton's initial velocity, we may set this component to 0 and only consider the proton's motion in the plane perpendicular to the wire.

We separate the proton's velocity into its radial component v_r and tangential component v_t. The key here is to consider angular momentum of the proton, which is then $L = mv_t r$, and the torque about the wire by the magnetic field is $\tau = F_B r = ev_t Br$. Since the electric field is radial, it contributes no torque. Hence, we have:

$$\tau = \frac{dL}{dt} \Rightarrow eBr\frac{dr}{dt} = \frac{dL}{dt}$$

Integrating both sides with respect to time, we have:

$$\frac{1}{2}eBr^2 - \frac{1}{2}eBR^2 = L - L_0 \Rightarrow m_p v_t r - L_0 \Rightarrow m_p v_t r - \frac{1}{2}eBR^2 = L_0 = \text{const.}$$

This also implies that it is impossible for the proton to reach infinity. When the proton is farthest away from the wire, we have $v_{r,f} = \frac{dr}{dt} = 0$, hence the final velocity is purely tangential, i.e. $v_f = v_{t,f}$.

Let the farthest distance the proton can reach be $r_f = 0.2\,\mathrm{m}$. From conservation of energy, we have:

$$\frac{1}{2}m_p v_i^2 + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{R} = \frac{1}{2}m_p v_f^2 + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{r_f}$$

$$\Rightarrow \frac{1}{2}m_p v_f^2 - \frac{1}{2}m_p(v_{t,i}^2 + v_{r,i}^2) = \frac{e\lambda}{2\pi\varepsilon_0}\ln\frac{r_f}{R} \tag{1}$$

Using the conserved quantity $m_p v_t r - \frac{1}{2}eBr^2$ we derived earlier:

$$m_p v_{t,i} R - \frac{1}{2}eBR^2 = m_p v_f r_f - \frac{1}{2}eBr_f^2 \tag{2}$$

From (1), to minimise the initial velocity $v_i = \sqrt{v_{t,i}^2 + v_{r,i}^2}$, we need to minimise v_f. If we assume that v_f is positive, then from (2), to minimise v_f, we need to minimise $v_{t,i}$. This is achieved when $v_i = v_{t,i}$ and when $v_{r,i}$ is negative. Letting $v_{r,i} = 0$ and solving equations (1) and (2) simultaneously, we have:

$$v_f \approx 483215\,\mathrm{m\,s^{-1}} \quad v_{t,i} \approx -470695\,\mathrm{m\,s^{-1}}$$

Our assumption that v_f is positive is satisfied, hence v_i has minimum magnitude

$$\boxed{471000\,\mathrm{m\,s^{-1}}}$$

and is purely tangential.

Problem 3.2 (SPhO 2009/2011)**.** A solenoid of length $2l$ has an inner radius R_1 and an outer radius R_2. The current through the solenoid is I. (i) Show that the magnetic flux density, B, at the center of the solenoid is

$$B = \kappa n I l \ln \frac{\alpha + \left(\alpha^2 + \beta^2\right)^{\frac{1}{2}}}{1 + \left(1 + \beta^2\right)^{\frac{1}{2}}}$$

where n is the number of turns per square meter and α and β are functions of R_1, R_2 and l. Write down the expression for α and β. State the value of κ. (ii) Show that the length of the wire is

$$L = nV = 2\pi n \left(\alpha^{m_1} - 1\right) \beta^{m_2} R_1^{m_3}$$

where V is the volume of the winding and m_1, m_2 and m_3 are exponents that need to be determined. State the value of m_1, m_2 and m_3. (iii) Show that the B field at the center of the solenoid can be written as

$$B = G \left(\frac{P\lambda\sigma}{R_1}\right)^{\frac{1}{2}}$$

where G depends on the geometry, P is the dissipated power, $\lambda = n\pi r^2$ is the filling factor or fraction of the coil cross section occupied by the conductor, r is the radius of the wire and σ is the conductivity.

Solution. (i) We treat the solenoid as a collection of current loops with minimum radius R_1 and maximum radius R_2. Consider one such current loop with radius R carrying current I. The magnetic field due to this ring at a distance x along its axis is

$$B_{\text{ring}} = \frac{\mu_0 I R^2}{2\left(R^2 + x^2\right)^{\frac{3}{2}}}$$

We can then sum up the magnetic field due to all such rings. We can approximate this summation as an integral to get

$$B = \int_{R_1}^{R_2} \int_{-l}^{l} \frac{\mu_0 I R^2}{2\left(R^2 + x^2\right)^{\frac{3}{2}}} n \,\mathrm{d}x \,\mathrm{d}R = \mu_0 n I \int_{R_1}^{R_2} \left(\int_{-l}^{l} \frac{R^2}{2\left(R^2 + x^2\right)^{\frac{3}{2}}} \,\mathrm{d}x \right) \mathrm{d}R$$

The inner integral can be evaluated by a trigonometric substitution to get

$$\int_{-l}^{l} \frac{R^2}{2\left(R^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{l}{\sqrt{R^2+l^2}}$$

so we have

$$B = \mu_0 n I l \int_{R_1}^{R_2} \frac{1}{\sqrt{R^2+l^2}}\,\mathrm{d}R$$

The resultant integral can then be evaluated by another trigonometric substitution to get

$$\int_{R_1}^{R_2} \frac{1}{\sqrt{R^2+l^2}}\,\mathrm{d}R = \ln\frac{R_2+\sqrt{R_2^2+l^2}}{R_1+\sqrt{R_1^2+l^2}}$$

so we finally have

$$B = \mu_0 n I l \ln\frac{R_2+\sqrt{R_2^2+l^2}}{R_1+\sqrt{R_1^2+l^2}} = \mu_0 n I l \ln\frac{\frac{R_2}{R_1}+\sqrt{\left(\frac{R_2}{R_1}\right)^2+\left(\frac{l}{R_1}\right)^2}}{1+\sqrt{1+\left(\frac{l}{R_1}\right)^2}}$$

i.e. $\alpha = \frac{R_2}{R_1}$, $\beta = \frac{l}{R_1}$ and $\kappa = \mu_0$.

(ii) The length of one wire loop is $2\pi R$, so the total length (approximation the sum as an integral again) is

$$L = \int_{R_1}^{R_2}\int_{-l}^{l} 2\pi R n\,\mathrm{d}x\,\mathrm{d}R = 2\pi n\left(R_2^2-R_1^2\right) = 2\pi n\left(\alpha^2-1\right)\beta R_1^3$$

i.e. $m_1 = 2$, $m_2 = 1$ and $m_3 = 3$.

(iii) We have that

$$P = I^2\frac{L}{\sigma A} = \frac{I^2 L}{\sigma\pi r^2} \quad\Longrightarrow\quad nIl = \sqrt{\frac{P\sigma n\pi r^2 l^2}{2\pi\left(\alpha^2-1\right)\beta R_1^3}} = \sqrt{\frac{P\lambda\sigma\beta}{2\pi\left(\alpha^2-1\right)R_1}}$$

Substituting into the expression for the magnetic field,

$$B = \mu_0\sqrt{\frac{P\lambda\sigma\beta}{2\pi\left(\alpha^2-1\right)R_1}}\ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}} = \left(\mu_0\sqrt{\frac{\beta}{2\pi\left(\alpha^2-1\right)}}\ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)\sqrt{\frac{P\lambda\sigma}{R_1}}$$

in the desired form.

Problem 3.3 (APhO 2017)**.** Problem T1. There is a reason that this problem is in this set.

Solution. Read the solutions here. The key thing to note in this problem is the analogy between the superfluid velocity and magnetic fields, and between vortex filaments and current. Armed with that analogy, the rest of the problem should be easier to approach.

Problem 3.4 (IPhO 2012)**.** Problem T1C. Set by Kalda so you know it's going to be good.

Solution. Read the solutions here. This is a very difficult problem (but it is very elegant) – don't be stressed if you couldn't solve it.